Bifurcations on the Sugarspace

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Abstract

This paper looks at Agent–Based Simulations (ABS) from a Dynamical Systems (DS) point of view. For this purpose, first, a comparison of ABSs and DSs models with respect to modelling and analysis is made. Since qualitative analysis of high–dimensional, non–linear ABSs is not feasible, this paper introduces a new analysis approach using the example of the sugarscape population model. In this method, the birth and reproduction rate are measured for every possible population size, N, which yields an approximate iteration map for the ABS. Plotting the accumulation $\Delta N(N)$, therefore, provides a global view on the simulation as a whole. This helps understanding the dynamic behaviour of complex systems, and it makes possible a bifurcation analysis for the sugarscape simulation. Possible applications of this method for different kinds of ABSs are also discussed.

1 Introduction

During the last years, the research of emergent phenomena became interesting to a large number of researchers from very different research areas. This is due to the fact that the real world is full of emergent structures [9, 16]. Even though there is no standardized definition of emergence, most researchers agree that emergence is some kind of macro–behaviour produced by a number of individuals and the mutual dependencies in between them.

ABS is based largely on this metaphor. Especially popular among social scientists [5, 6], it represents one of the main approaches to understand emergent phenomena. In ABSs, a natural or social system is virtually created by placing a number of individuals in an environment, and by defining simple rules of interaction in between the individuals, as well as in between the individuals and their environment. In the analysis of ABSs, usually, aggregate variables are measured and displayed.

A second approach to understand complex real–world phenomena uses DS theory and mathematical modelling techniques [8, 10, 1]. It bases on the finding in the natural sciences that many natural phenomena are determined by differential equations (DEs), and, in fact, DS theory developed from the theory of DEs. Here, models are developed that match the macro–behaviour observed in the real world. These models are analysed by numerical simulations as well as by analytical considerations, and it is often possible to understand completely how such a system behaves as $t \to \infty$.

Since ABSs and DSs have been used in the research of very related phenomena, combining the two can bring along a better understanding of emergent behaviour. In particular, a more formal treatment of ABSs becomes possible when looking at these simulations from a qualitative viewpoint.

In this paper, a graphical method useful in the analysis of DSs is applied to ABSs. This concept is illustrated at the sugarscape population model presented in [5], and it is shown to be possible to qualitatively understand the dynamics of this simulation. Computing the expected change in the aggregate variable (population size, N, in this case) of the ABS for all possible N, yields an approximate iteration map $N^{t+1} = f(N^t) = N^t + \Delta N^t$, the analysis of which is the main business of the DS theory. A qualitative view on the simulation as a whole can then be obtained by plotting the $N - \Delta N$ diagram. Generating these plots for different parameters in the agent rules, it is even possible to detect bifurcations in the sugarscape model.

The remainder of this paper is organized as follows: The next section presents the research related to the present discussion. Then, positive and negative aspects of ABS and DS are discussed with respect to the modelling and the analysis task. In Section 4, after introducing the graphical method, a qualitative analysis of the sugarscape model is performed. Finally, the results are discussed and a conclusion on the paper as a whole is drawn.

2 Background and Related Work

Although there are some works pointing out the need for a joint research of ABSs and DSs (e.g., [5, 11, 15]), there are only a few attempts to combine them in the analysis. Presented papers compare the two approaches (as [15] for cellular receptor dynamics and [14] for bilingualism in a small population), but a direct application of DS analysis methods to AB systems is usually not done.

The author found only three attempts which go beyond the comparison of both approaches [7, 3, 4]. Fahse et al. [7] and Duboz et al. [3], use ABSs to make mathematical population models more realistic. The population growth rate is extracted from an ABS, in which the agents do behave, but do not age, die and reproduce. A similar approach is used here in the estimation of the iteration map for the sugarspace population. However, in [7] and [3], the extracted data is used to adjust the parameters of a mathematical model. Qualitative analysis can be performed using this model. In this paper, a way to directly analyse ABSs, without deriving a global, aggregate model from it, is presented.

In [4], Edwards et al. compare an individual–based model for innovation diffusion with an aggregate model, derived from the AB model using the probability distribution in the model states. They show that the aggregate model approximates the results of the AB model in some cases, but fail to do so in other cases. Graphical analysis of the aggregate model is used to explain the differences in the results. The graphical method used in [4] is very similar to the method used in this paper. However, as noted above, we are interested in understanding the dynamics of ABSs using data directly obtained from the simulation.

For this purpose, an AB population model based on the sugarscape model presented by J. M. Epstein and R. Axtell [5] is used. Even though the sugarscape example is well–known among social scientists, it seems necessary to the author to recall the most important components of this model.

2.1 The Sugarscape Population Model

The sugarscape model, agents act in an environment consisting of 50×50 fields, each of which holds a certain amount of sugar. During each simulation step, the agents move in the environment, collect sugar and reproduce if certain conditions are fulfilled. Agents die, if their age (incremented in each step) is above an initial (randomly chosen) value between 60 and 100, or if they have no sugar. This can happen, because agents consume a certain amount of sugar defined by their metabolism. The agent's metabolism is uniformly distributed in $\{1, \ldots, 4\}$. Along with differing vision from agent to agent (uniformly in $\{1, \ldots, 6\}$) and a different amount of initial sugar (uniformly in $\{50, \ldots, 100\}$), this results in a heterogeneous agent population acting on the sugarspace.

For the sake of comparability[1], the implementation used in this paper relies as much as possible on the implementation described in [5]. The sugar grow back rule $G(\alpha)$ is defined by $s^{t+1} = min(s^t + \alpha, s_{max})$, which means that, in each time step, the amount of sugar, s, in each lattice point is increased by α until the maximum sugar level s_{max} is reached. The agents are allowed to move ($M(v)$) in the four major directions, as far as vision allows. They search for the nearest unoccupied position of maximum sugar, go there and collect the sugar.

After moving, agents attempt to reproduce with all their neighbours in the Moore neighbourhood, choosing them in random order. But reproduction is only allowed if both agents are fertile, of opposite sex, and if there is a free field near one of them (the "child" is placed at this field). Each time reproduction takes place, the agent's wealth (the sugar it accumulated) is divided by two. A male agent is fertile if his age is between a and b, and if his accumulated sugar exceeds e. A female agent is fertile if her age is between c and d, and if her accumulated sugar exceeds e. The reproduction rule can therefore be formally written as $R(a, b, c, d, e)$.

3 AB Modelling and DS Analysis

ABS and DS are both concerned with the modelling and the analysis of complex social, biological, etc. phenomena, in order to understand better the behaviour observed in the real world. In general, however, they differ greatly in the way they approach the modelling and the analysis task.

In ABSs, a number of individual agents is distributed in a virtual space, and the modelling is mainly concerned with defining appropriate rules of interaction between these individuals. Rules usually include stochastic processes, accounting for heterogeneous individual behaviour. And also, in most of the cases, the initial agent distribution in space is determined by stochastical means. Once the rules are specified, the behaviour of the system is obtained by running the simulation.

In contrast to ABS, DS modelling starts at the global, more abstract level. Usually, time series data, observed in the real world, is used as a starting point for the development of mathematical models[2]. The modeller uses (sometimes very specific) knowledge of the respective research area, to create a prototype model that includes the most important aspects of the considered problem. The parameters of such a prototype model are then adjusted, in order to fit the data found in the real world system. By looking at the problem from a global viewpoint, DSs do not take into account individual aspects, implicitly assuming the population to be homogeneous. This can be a reasonable assumption in some cases, but

[1] One problem of research using ABS is the difficulty for others to reproduce and check simulation results. In general, two implementations of the same problem will differ to some extend, and so will the simulation results.

[2] See, e.g., [10] and [12] for good overviews of mathematical population models.

there are other cases where heterogeneity plays an important role.

Besides the presence of heterogeneity, the main benefit of AB modelling is that these models can be heuristically understood by the non–scientist. Rather intuitively, observed individual behaviour is put into simple agent–agent rules, and that's all about it. The DS approach, in contrast, is often clear only to the specialist, since the behaviour of the individuals, though implicitly considered during the modelling, is not obvious in the final model. As a result, DS mathematical models are often difficult to communicate to the non–professional.

But the DS approach bears a great advantage over the AB approach: it can be analysed by qualitative means, so that, in a considerable number of cases, the dynamics of the model can be completely understood. By such a qualitative analysis, it is possible to assure what happens to the system as time approaches infinity, $t \to \infty$. Moreover, it is possible to find exact parameter constellations, at which the system dynamics change from one behaviour to another. In the theory, this is referred to as a bifurcation [2, 8].

Detecting such qualitative changes of the dynamics resulting from ABSs would bring along great advances in the understanding of these models, because specific changes in the rules of the model could be related to specific changes in the emergent structure. This would facilitate the modelling process, because the qualitative information can be used to find rules that yield realistic behaviour. Moreover, a better understanding of systems of multiple mutual dependent individuals can be gained, once the rules between the individuals can be related to specific dynamic behaviours of the system. The next section describes a possible way of detecting qualitative changes of the sugarscape ABS.

By now, analysis of ABSs is usually performed by running a series of simulations, while, more or less systematically, changing the initial setup and the parameters of specific rules. This is a tedious and often tremendously time–consuming task. Moreover, the results have to be examined very carefully, since, due to the stochastic processes involved, no two simulations yield the same result. Furthermore, we cannot be absolutely sure how the system evolves as $t \to \infty$, because each simulation run is stopped after a particular time.

The latter question, however, is also important in DS research, if numerical simulations are carried out to analyse the system behaviour. In general, numerical analysis, considering a finite time interval only, does not provide assured knowledge about the long–time behaviour of the system. This becomes evident having in mind that we are dealing with dynamics at the edge of chaos and order, in which case small changes in the simulation configuration (or even the limited precision of floating point operations!) might cause tremendous differences in the simulation result. In DS research as well as in ABSs, it is therefore very important to look at the system from an analytical viewpoint, since assurance of the system behaviour can be obtained only by this means.

To summarize, ABS is well–suited for the modelling of complex systems, but a complete analysis of the system behaviour can be very difficult (often impossible). DS theory, on the other hand, provides a useful framework for the analysis of complex systems, but simplifying assumptions for the modelling might make mathematical models less realistic and less understandable. Applying DS methods to ABSs, as shown in the sequel, is one way to overcome these problems.

4 Population Dynamics on the Sugarspace

The research of population dynamics aims to explain how the number of individuals of a certain species evolves in time. We are not interested in how a particular individual behaves. Therefore, often mathematical models have been used in this area [10, 12]. However, Epstein and Axtell [5] showed that realistic population dynamics can be observed also in an ABS, in which the heterogeneity of the individuals is explicitly taken into account. In this section, a formal analysis of the dynamics of their sugarscape population model, defined by the rules presented in Section 2.1, is performed.

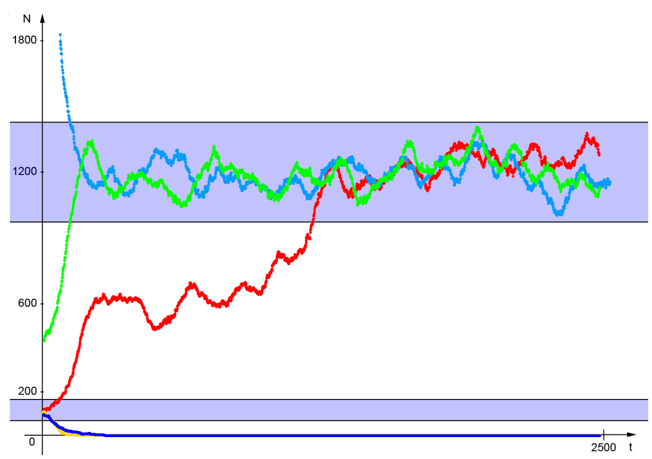

Figure 1: The sugarscape population size for 2500 time steps. Different colours represent different initial number of agents, N_0.

Let's first look at different time series (also called orbits) produced by that model, when altering the initial number of agents N_0. The grow back rule $G(3)$, and the reproduction rule $R(a, b, c, d, 50)$ with $a, c \in$

$\{12, \dots, 15\}$, $b \in \{40, \dots, 50\}$ and $d \in \{30, \dots, 40\}$ are used[3]. In Figure 1, basically, two possible courses are observed, depending on the initial number of agents. If the initial population is small ($N_0 < 80$), an extinction of the species is likely to happen (dark blue and yellow), and if N_0 is large enough (>180) the population cycles around $N = 1200$ (blue, green and red). In the case that $80 < N_0 < 180$, both behaviours are possible, which is due to the stochastic parameters involved into the simulation.

From a qualitative viewpoint, there are three important points (regions). Clearly, $N_1^* = 0$ represents a fixed point attractor, for once this point is reached, the orbit always remains there[4]. Another attractor is represented by the upper blue shaded region around $N = 1200$. In all the simulations performed, the population remained within this region, once it was reached. The second blue shaded region at $80 < N < 180$ represents a repelling region [13]. Orbits that start in this region will eventually leave it under iterations.

The number of agents on the sugarscape is a one–dimensional (1D) variable. Therefore, the time evolution of N can be written as a iteration map $N^{t+1} = f(N^t)$. In 1D DSs, where f is usually known explicitly, the behaviour of the system can be understood by a graphical analysis[5]. For ABSs, however, we do usually not have at hand an explicit map, for the transition from one time step to the other is defined by a high–dimensional, non–linear and stochastic system. But by measuring the deciding variables of the ABS, we can obtain an approximate map $f(N^t)$, which allows us to use similar methods.

The evolution of the sugarscape population from one time step to the other is defined by

$$N^{t+1} = N^t + r(N^t) - d_h(N^t) - d_a(N^t), \qquad (1)$$

where $r(N^t)$ represents the number of new born agents, $d_h(N^t)$ the number of agents dying, because they lack sugar, and $d_a(N^t)$ accounts for the number of agents that die, because they reached their maximum age. A map f can therefore be obtained by measuring those three terms.

This measurement is done by stepwisely increasing the number of agents, N, from zero to 2500 ($= 50 \times 50$), and by counting the new born agents and the agents that would die. Note that we do not remove dying agents, nor do we create the new borns; we merely count the cases in which the conditions for the respective action are fulfilled. Because results will differ for each run, the average of a number of time steps is taken. This number should be larger than 100 in order to make sure that at least one entire generation of agents is considered in the measuring. Here, 111 time steps have been used.

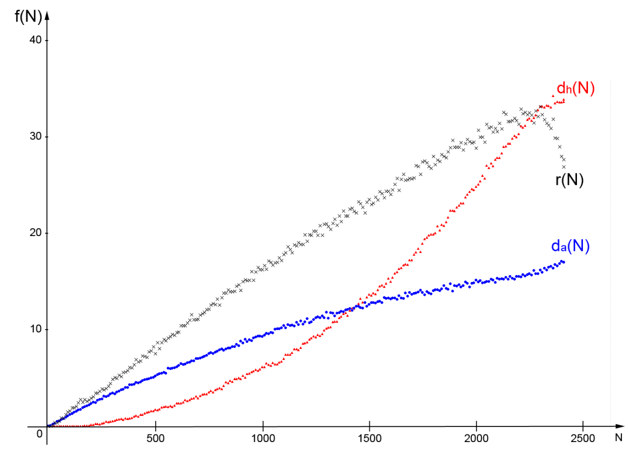

Figure 2: Measuring $r(N), d_h(N)$ and $d_a(N)$ for all N to obtain an approximate iteration map $f(N)$.

The measurement is started with $N = 0$. Over the next 111 simulation steps, r, d_h and d_a are summed up and divided by 111 to obtain the average reproduction and death rate. (Obviously, $r(0) = d_h(0) = d_a(0) = 0$.) Then, 10 new agents are placed randomly on the sugarspace, and the counting procedure is repeated for the estimation of $r(10), d_h(10)$ and $d_a(10)$. After another 111 time steps have passed, again, 10 more agents are created and the measurement is performed. Repeating this procedure for $N = 0, 10, 20, \dots, 2500$ yields an estimate of the unknown terms in 1, as shown in Figure 2.

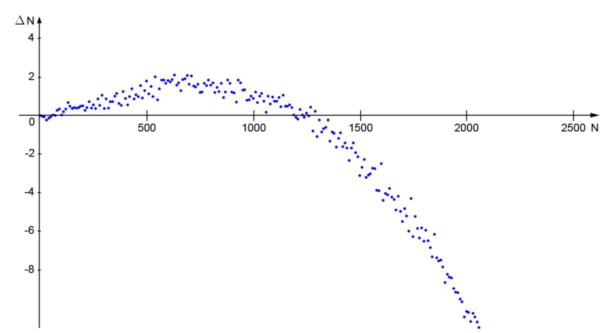

Figure 3: ΔN is plotted with respect to N. Since the data is obtained from a stochastic ABS, $\Delta N(N)$ alters within some range.

Having these estimates, we plot the change in the agent number defined by $\Delta N = r(N) - d_h(N) - d_a(N)$

[3] The reproduction rule corresponds to the rule used in [5] (page 64). However, there, the grow back rule $G(1)$ is used.

[4] A fixed point N^* is a point for which $N^* = f(N^*)$. See [2, 8] for details.

[5] The reader is referred to [2] for an excellent description of graphical analysis methods.

for all N. This is shown in Figure 3. By the help of this plot, we see for which N the agent number increases ($\Delta N > 0$) and for which N a decrease happens ($\Delta N < 0$). The N for which $\Delta N \approx 0$, represent the attracting and repelling regions of the simulation, since the agent number is not changing in these cases.

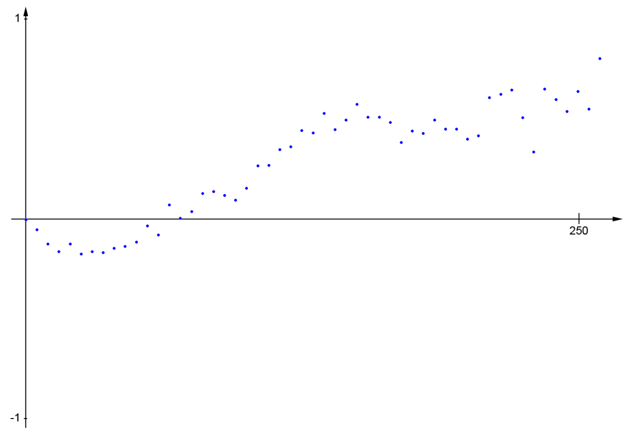

Figure 4: A close up view on $\Delta N(N)$ for $0 < N < 250$.

Figure 3 shows the course of $\Delta N(N)$ for $0 < N < 2500$. Since the course near $N = 0$ is not clearly visible at this scale, a close up view is shown in Figure 4. The images illustrate that there are three regions in which $\Delta N(N)$ crosses the horizontal axis, for $G(3)$ and $R(a, b, c, d, 50)$. And clearly, the values N at which this happens, correspond to the three decisive regions pointed out in Figure 1. The first crossing takes place at $N_1^* = 0$, which verifies that this is a fixed point. Looking at the estimates, ΔN, near this point makes clear that this point is attracting. Since values are below zero, $N^{t+1} < N^t$, and the series approaches $N_1^* = 0$ after a few iterations.

The second crossing region is located around $80 < N < 140$. Let's denote this by $N_2^* \approx \{N : 80 < N < 140\}$. This region is repelling, since $\Delta N < 0$ for $N < N_2^*$ and $\Delta N > 0$ for $N > N_2^*$. This means that the iteration of smaller N eventually approaches zero, and that also the iteration of larger N forces the series away from this region. In the latter case, the orbits will eventually reach the second attractor of the map, $N_3^* \approx \{N : 1100 < N < 1400\}$. This region is attracting, since $\Delta N > 0$ for $N < N_3^*$ and $\Delta N < 0$ for $N > N_3^*$. In particular, it becomes obvious that very large initial populations diminish under the repeated iteration of the ABS until they reach N_3^*.

The main benefit of looking at the ABS in this way, is that it provides a global view on the simulation as a whole. From a single graph, we obtain knowledge of the system behaviour for all possible initial values N_0. Moreover, to a certain extend, it provides the means to assure the systems behaviour as $t \to \infty$. For instance, since ΔN is significantly larger than zero for $500 < N < 1000$, the sugarscape population will not die out once it reached N_3^*.

However, also some care must be taken in the interpretation of such diagrams. Since ΔN is a stochastic term and an average value is used for its estimation, there might be simulation steps for which the actual ΔN departs greatly from the estimation. Therefore, a critical consideration of the significance of the estimated ΔN is necessary, mainly regarding the distance to $\Delta N = 0$ and the variance of the estimation[6].

At any rate, having at hand a global overview of the system dynamics, facilitates a bifurcation analysis for the sugarscape simulation. Basically, a bifurcation indicates a qualitative change of the system behaviour[7]. Plotting the $N - \Delta N$ diagrams for different parameters in the rules of the ABS, we may find out for which parameters such a qualitative change happens.

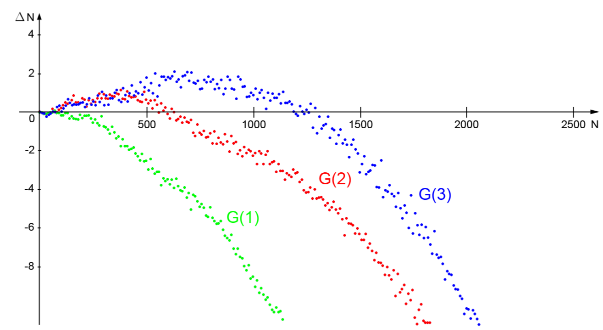

Figure 5: The $N - \Delta N$ plot for different grow back rules.

Figure 5 shows that at the example of the sugar grow back rule $G(\alpha)$. The $N - \Delta N$ diagram is generated for $\alpha = 1, 2, 3$. In the case of $G(3)$ and $G(2)$, three regions for which $N \approx 0$ are observed. If $G(1)$ is used, whereas, $\Delta N(N)$ shows a monotonically decreasing course, which is zero only at $N = 0$. This indicates that a bifurcation takes place between $G(2)$ and $G(1)$. In fact, the ABS using $G(1)$ does not any longer possess the regions N_2^* and N_3^*, but a single fixed point at $N_1^* = 0$, to which all the orbits are attracted. This is a qualitative change in the system's behaviour.

The Figures 6 and 7 show the respective time sequences for $G(2)$ and $G(3)$ using a number of different

[6] For instance, the values for $0 < N < 80$ are very close to zero, but the variance is also low in this region, whereas for $500 < N < 1000$ the variance is larger, but so is the distance to the zero axis.

[7] The definition of a bifurcation in DS theory is based on the notion of structural stability. Since the description of this concept for ABSs is beyond the scope of this paper, the reader is referred to [8] for a very clear definition of a bifurcation in DSs.

initial values. This makes clear that it is indeed possible to understand the behaviour of the sugarscape simulation, by means of a single diagram.

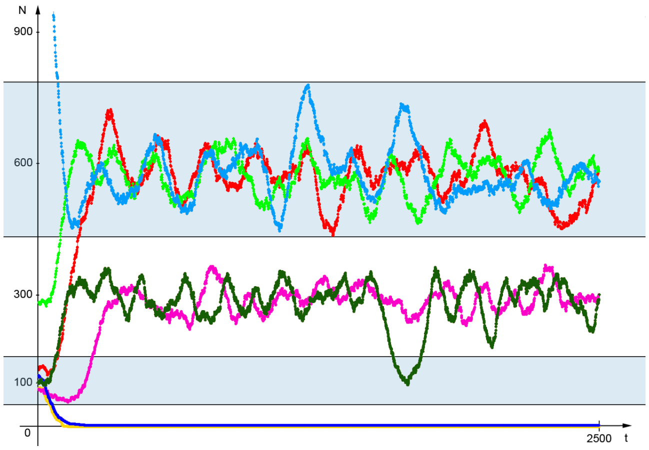

Figure 6: Different orbits of the ABS using $G(2)$. Note the surprising course of the dark green and magenta orbit. In both cases, the agent population dies out at one of the sugar mountains, but settles on the other one.

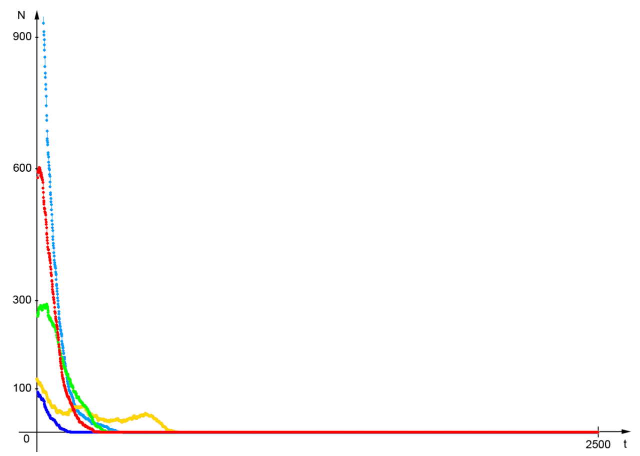

Figure 7: Different orbits of the ABS using $G(1)$. An extinction of the population takes place in all the cases.

But in Figure 6, also the limitations of graphical analysis of ABSs become clearer. First, a fairly surprising behaviour can be observed for orbits starting at the repelling region $N_2^* \approx \{N : 50 < N < 130\}$: in some cases, the agent population dies out at one of the sugar mountains, but settles on the other one. This indicates the existence of a third attracting region for this system. Unfortunately, this behaviour is not obvious from the $N - \Delta N$ plot. (Note, however, that for $G(3)$ this attractor does not exist, since agents are able to re–settle on the other sugar mountain.) Another limitation of the method becomes clear by looking at the great outburst of the dark green orbit starting in the repelling region. The population size returns to N_2^*, in which case an extinction of the population is also possible. A more detailed graphical analysis of the region around $N = 300$ could bring along a deeper understanding of this effect.

Let's consider another bifurcation, which takes place when $G(1)$ is used and the reproduction rule $R(a, b, c, d, e)$ is changed. As shown in [5], the most crucial parameter in this rule is the amount of sugar required for the agents to be fertile, e. While $e = 50$ results in extinction of the population in all the cases (compare Figure 1), $e = 28$ yields an orbit cycling around $N = 2000$ (see Figure 8 and 9). Obviously, a qualitative change of the system dynamics takes place.

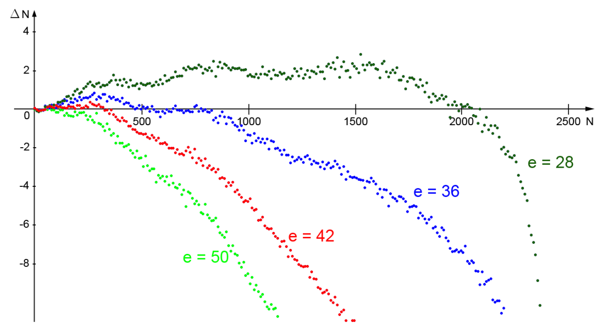

Figure 8: The $N - \Delta N$ plot for different values e in the reproduction rule $R(a, b, c, d, e)$.

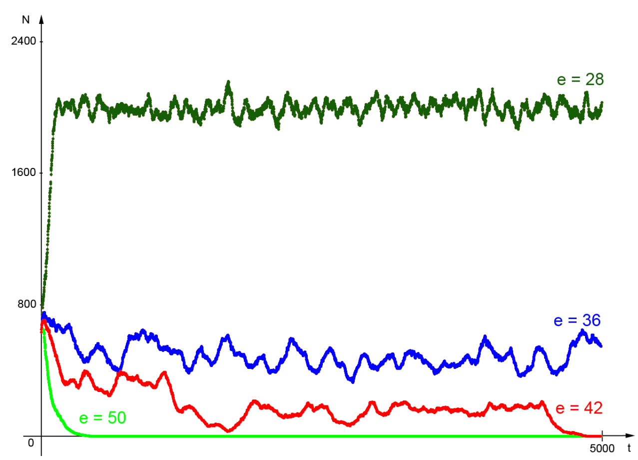

Figure 9: Time sequences obtained by different reproduction rules using the same initial population N_0. Note that 5000 iterations are used here.

Figure 8 compares the $N - \Delta N$ plots for four different values e. For $e = 28$ and $e = 36$, just as in the case of

$G(2)$ and $G(3)$ in Figure 5, three regions are observed in which $\Delta N(N)$ crosses the horizontal axis. Therefore, from the qualitative viewpoint, the same behaviour is to be expected, i.e., the system possesses two attractors: the trivial fixed point $N_1^* = 0$ and N_3^*, and also an repelling region N_2^*. For $e > 42$, the system only possesses one attracting fixed point, $N_1^* = 0$, which means an extinction of the population in every case. But what happens in the case that $36 < e \leq 42$?

To answer this question, let's consider for a moment that $\Delta N(N)$ is a deterministic function and e is a continuous parameter. In this case, there is a parameter value $\tilde{e}$ for which exactly two fixed points N with $\Delta N(N) = 0$ exist, one of which is given by $N_1^* = 0$. In the left and right neighbourhood of the second fixed point, N_2^*, ΔN is below zero. For this reason, this point attracts orbits starting at $N > N_2^*$, but iterations of $N < N_2^*$ are repelled by it and tend to zero.

Although we are not dealing with deterministic systems, this reasoning can be applied to the example considered here. Actually, the effect of a region attracting to the one side and repelling to the other is observed in the time series produced by $e = 42$ (see Figure 9). The orbit first approaches an attractor, remains there for some (relatively long) period, but eventually leaves the region and approaches $N_1^* = 0$. Obviously, this is due to the stochastic parameters involved into the ABS. On a long term, it is likely that a sequence of random values is produced, which brings the population size very close to N_1^*, so that it is eventually attracted by this point.

And randomness is also the reason for which it is impossible to find an exact parameter value $\tilde{e}$, at which the system undergoes this bifurcation. However, close–up analysis of the critical region around N_2^* indicates that $39 \leq \tilde{e} \leq 42$. Recalling that the sugarscape model is highly non–linear, mostly discrete in the state variables and stochastic, the author considers this a useful approximation.

5 Discussion

Using the graphical method presented in the last section, the researcher is given at hand a single diagram to predict and explain the behaviour of an ABSs. This can help greatly in the analysis of such systems.

Moreover, providing a global view on the simulation as a whole, the method can be used to compare different implementations of the same problem. By this means, the results of ABSs become more traceable by others. For instance, in the implementation of the sugarscape model used here, no reproduction rules $R(a, b, c, d, e)$ have been found that produce regular, large amplitude oscillations as shown in [5] (pages 65 and 66). Sequences move less regular and remain within the attracting region. The reasons for the differences in the behaviour could be found by comparing the $N - \Delta N$ diagram for both implementations.

However, in order to become a general analysis method for AB models, its applicability to different kinds of ABSs must be shown. Obviously, when ABSs are used to simulate the time–behaviour of a single aggregate variable, the application of the method is straightforward. Basically, an application is possible in all the areas in which, traditionally, ABSs and mathematical models are used. For different kinds of ABSs, it is necessary to find a global variable which provides information about the important states of a system, and to use estimates of this variable in the generation of the $N - \Delta N$ diagram. This can be difficult in some cases (e.g., if the emergent structures are spacial patterns as in Conways Game of Live), but relatively easy in others.

The last issue to be discussed here concerns stochasticity. Obviously, the presented method is an approximate one, because the estimate of f relies on very complex simulation which involves a large number of stochastic parameters. Therefore, care must be taken in the interpretation of the results, especially if $\Delta N(N)$ is close to zero in a bigger range. In this case, it is theoretically possible that a sequence of random numbers is produced, so that an orbit which entered an attractor might eventually leave the region again. Although generating large–scale $N - \Delta N$ diagrams for the problematic region can bring along better understanding of this effect, a real solution to the problem is to find adequate stability measures based on probability theory.

6 Conclusions and Future Work

This paper presented a graphical method for the formal analysis of ABSs at the example of the sugarscape population model. The method provides a global view of the system behaviour, which allows to predict and understand the possible outcome of the simulation. It also enables a bifurcation analysis for ABSs, by means of which particular parameters in the interaction rules can be related to particular dynamics of the model.

The paper therefore brings closer DS theory and ABSs, for it shows that graphical tools often used in the analysis of 1D discrete DSs can be applied to ABSs. The key to such an analysis is the measurement of aggregate variables of the simulation: the reproduction rate and the death rate in the used population example. By this means, it is possible to derive a approximate iteration map of the form $N^{t+1} = f(N^t)$, representing the time–evolution of the sugarscape population. This map is graphically displayed in a $N - \Delta N$ diagram, with the result that the expected change from one iteration to the next one, ΔN, is visible for all possible population numbers. Straightforwardly, it is possible to find out the equilibrium states (attractors) of the ABS, since these regions are defined by $\Delta N(N) \approx 0$. Considering $\Delta N(N)$ in the neighbourhood of those regions, makes clear whether a region is attracting or repelling. Therefore, the diagram shows for which initial values which equilibrium state is reached.

Comparing the $N - \Delta N$ diagrams generated for different parameters in the rules of the ABS, facilitates the bifurcation analysis of ABSs. This was shown for the environmental sugar grow back rule $G(\alpha)$, as well as for the reproduction rule $R(a, b, c, d, e)$ with respect to e. Although it is in general not possible to find an exact bifurcation value $\tilde{e}$ (for it does not exist, if stochastic processes are involved), reasonable upper and lower bounds of $\tilde{e}$ can be derived. The bifurcation analysis of ABSs is an important step to understanding completely the dynamics which possibly emerge from complex simulations, and therefore, also a deeper insight into emergent phenomena can be gained.

There are several issues to be addressed by future research, the most important of which is the application of the method to different kinds of ABSs. As pointed out in the previous section, this is expected to be feasible in a considerable number of cases, but might be difficult in other ones. Another important task is the use of other DS tools in the analysis. Especially the questions concerning the dynamics within an attracting region, as well as the actual stability of a detected attractor have to be considered in the future. Knowing the probability of orbits to leave an attractor, because of the stochastic values of the ABS, would allow more definite knowledge of the system behaviour as $t \to \infty$. Once such analysis means are established, a qualitative analysis of ABSs comparing to the analysis of DSs will become possible.

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